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LETTER TO THE EDITOR

The 'active perimeter' in cluster growth models: a rigorous bound

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Abstract. A definition of an 'active perimeter' is introduced for arbitrary cluster growth models. A general bound is derived, showing that this perimeter cannot grow faster than R^{d_f-1}, where d_f is the fractal dimension of the cluster. For diffusion-limited aggregates, this is seen to be smaller than the 'growing interface' as defined by Meakin and Witten. A further consequence is that the Eden model is fully compact in any dimension d, i.e. it has a negligible number of internal holes. Further one finds that its perimeter scales as $(\text{volume})^{(d-1)/d}$

Models for cluster growth have recently attracted a great deal of interest, in particular due to their ability to generate complex, highly ramified, 'fractal' structures (for references see, e.g., Family and Landau 1984). For such a phenomenon to occur, it is necessary to have some form of screening, so that the growth process does not eventually fill in all the large-scale voids present. As a consequence, attention has focused on the so-called growing interface (Meakin and Witten 1983), i.e., that part of the cluster where growth principally occurs. Meakin and Witten (1983), in particular, have given the following definition of the growing interface for diffusion-limited aggregation. Let first a cluster of N sites be grown, after which it is further grown for an arbitrarily long time. The growing interface is the set of these points put down in the first N steps that are adjacent to at least one point put down after the first N steps.

It is the purpose of this letter to introduce another definition of 'active perimeter', for which rigorous bounds can then be derived. It will be seen that the 'active perimeter' of diffusion-limited aggregates (DLA) is much smaller than the interface defined by Meakin and Witten (1983). However, for models such as the Eden model (Eden 1961, Sawada *et al* 1982, Rikvold 1982), this 'active perimeter' actually scales as the total perimeter.

A cluster growth model starts as a fixed cluster, typically a seed, to which adjacent sites are added one at a time according to certain probabilities. If x denotes a point adjacent to the cluster at time N (i.e., when the cluster has N sites), then the (normalised) probability that x will be added to the cluster at time $N+1$ is defined to be $\pi_N(x)$. For the case of fractal growth it is well known (Turkevich and Scher 1985, Halsey *et al* 1985), that the growth probabilities $\pi_N(x)$ vary sharply as a function of position. In particular, only a very small portion of sites have a growth probability comparable to the maximum growth probability. To obtain an approximate measure of the number of such sites, let the active perimeter of a growth model be defined as follows: give each site x a weight measuring its growth probability with respect to the maximum growth probability. The active perimeter at time N, P_N, is then the sum of these weights, or put differently:

$$P_N = (\max_x \pi_N(x))^{-1} = \sum_x \frac{\pi_N(x)}{\max_y \pi_N(y)}. \tag{1}$$

The fundamental result I wish to derive is the following: for *any* cluster growth process, for N sufficiently large:

$$P_N / N \leqslant C'' / R_N, \tag{2}$$

where R_N is a length of the order of the radius of gyration of the cluster at time N and C'' is a constant.

To prove this, consider the convex hull of the cluster, i.e., the smallest convex set containing the whole cluster. To construct this set, connect every point on the cluster with every other by a straight line (or by the corresponding set of points if the cluster is on a lattice). The resultant set is not a fractal, but a convex set bounded by a polyhedron. It has, therefore, a well defined area and volume. Let V_N be the volume of the convex hull at time N, and let S_N be its surface. Since the convex hull has no interior holes and is bounded by a regular surface, one has $S_N \propto R_N^{d-1}$ and $V_N \propto R_N^d$, where R_N is a typical cluster radius. This assumes, of course, that there are no two directions growing at infinitely different rates, since in this case one could not express S_N and V_N in terms of one length only.

For any cluster point x on the boundary of the convex hull, define $S(x)$ as the surface of all $(d-1)$-dimensional faces of the hull adjacent to the vertex x. If a is the lattice spacing, then clearly the increase in volume due to the cluster growing at x from time N to time $N+1$ is

$$\Delta V(x) \propto aS(x),$$

and hence the average rate at which V_N increases, summing over all vertices x on the boundary of the convex hull, is given by

$$\left\langle \frac{\mathrm{d} V_N}{\mathrm{d} N} \right\rangle \propto a \sum_x \pi_N(x) S(x) \leqslant C S_N \max_x \pi_N(x) = C S_N / P_N, \tag{3}$$

where C is a constant, containing a factor of d to account for multiple counting of $(d-1)$-dimensional faces on the sum over x. This now leads to

$$\left\langle \frac{\mathrm{d} V_N / V_N}{\mathrm{d} N / N} \right\rangle \leqslant \frac{C S_N / V_N}{P_N / N}. \tag{4}$$

But since one has

$$N \leqslant V_N \leqslant \text{constant } N^d, \tag{5}$$

it follows that, taking R_N to be V_N / S_N, as mentioned above:

$$\ln N \leqslant \ln V_N = \sum_{M=1}^{N} \frac{\mathrm{d} \ln V_M}{\mathrm{d} M} \leqslant C' \sum_{M=1}^{N} \frac{1}{R_M P_M}, \tag{6a}$$

or, if R_N and P_N have a well defined asymptotic power-law behaviour

$$P_N / N \leqslant C'' / R_N. \tag{6b}$$

Now consider some of the consequences of this inequality. The (site) Eden model is characterised by the fact that all $\pi_N(x)$ are identical, i.e. they do not depend on x. From this follows immediately that P_N is in fact proportional to the total (external and internal) perimeter. This is, however, equally true of a considerable class of other models (Rikvold 1982, Sawada *et al* 1982, Meakin 1983) some of which appear fractal over a large range of length scales. For all these models it is readily seen that $\pi_N(x)$ never varies by more than a constant factor, i.e.

$$0 < C_1 \leqslant \pi_N(x)/\pi_N(x') \leqslant C_2, \tag{7}$$

for any N, x and x'. From inequality (6) it follows immediately that the Eden model and its variants are compact in any dimension (Richardson 1973), i.e. it does not contain a finite fraction of voids. This follows from the inequality, since only a vanishingly small portion of sites are connected to an empty site. This, however, leads to $N \propto R^d$ and hence, again by inequality (6)

$$P_N \leqslant C R_N^{d-1} \tag{8}$$

which implies $P_N \propto R_N^{d-1}$, since the total perimeter cannot be less than constant R^{d-1}. Further, it is clear that the external perimeter, say, cannot be a fractal object, since it scales at the most as R^{d-1}.

For diffusion-limited aggregates (Witten and Sander 1981), P_N defines a new concept of 'active perimeter'. Inequality (6) can be rewritten as

$$P_N \leqslant C'' R_N^{d_f-1}, \tag{9}$$

where d_f is the fractal dimension of the aggregate. Since no site on the boundary of the convex hull can screen another site on the boundary of the convex hull from the diffusing field, one can assume all $\pi_N(x)$ to be comparable for x on the convex hull. This implies that P_N actually scales exactly as R^{d_f-1}, as was independently pointed out by Turkevich and Scher (1985).

The growing interface as defined by Meakin and Witten (1983), however, scales as R^{d-1}—where d is the space dimension—for $d = 2, 3$. Analytical work (Carleson 1985) seems to confirm this result for $d = 2$. This means that the active perimeter as defined above is a vanishingly small portion of the growing interface. This indicates that among all of those original sites at which growth will occur at some later time, there are only very few which have a growth probability comparable to the maximal growth probability. This is a confirmation of the empirically well known fact that a small minority of points is most active during growth, while the others' growth is quickly stunted. This result may be compared to the appearance of 'forgotten growth sites' in the growth models studied by Bunde *et al* (1985b).

It may be worthwhile to point out that the inequality derived by Ball and Witten (1984) can be rederived using the above inequality. This is hardly surprising, since the argument involved in both proofs is very similar. For this purpose, it is sufficient to show that a walker has a finite probability of hitting a site on the boundary of the convex hull, since this implies, in the general case where the walk has dimension d_w:

$$d_f - 1 \geqslant d - d_w \tag{10}$$

which is the desired result. To prove this, we remark that the probability of hitting a site on the boundary of the convex hull is roughly the number of sites on this boundary multiplied by the maximal growth probability, since all these sites are unscreened. But the number of cluster sites on a regular surface scales as R^{d_f-1} and the maximal growth probability scales as $R^{-(d_f-1)}$, thus completing the proof.

For percolation clusters a result can be obtained as follows: a percolation cluster can be grown by starting from a seed, choosing a perimeter site at random and then occupying it with probability p_c and blocking it with probability $1-p_c$. This algorithm—together with variations—was proposed originally by Leath (1976) and Alexandrowicz (1980). As a growth process it has been extensively studied by Bunde *et al* (1985a). Clearly P_N for this algorithm corresponds to the number of 'growth sites' G as defined by Bunde *et al* (1985a), i.e., the number of occupied sites adjacent to an empty unblocked site. Defining d_G by

$$G \propto R^{d_G},$$

it follows from (6) that

$$d_G \leqslant d_f - 1, \tag{11}$$

where d_f is the fractal dimension of percolation clusters. This is confirmed by the numerical results of Bunde *et al* (1985a) in two dimensions, where $d_G = 0.76 \pm 0.02$ with $d_f = 1.89$. The inequality (11) can, however, be violated if correlations are introduced, as was also investigated by Bunde *et al* (1985a). For example, in the case where the probabilities $\pi_N(x)$ depend upon the distance between x and the site chosen at time $N-1$, if short distances are strongly preferred, the large probabilities are entirely concentrated on a very small set, so that P_N is of order one independently of N. The inequality (6) then becomes vacuous and inequality (11) does indeed fail.

Summarising, a definition for the active perimeter of an arbitrary cluster growth process has been given and a general upper bound for it has been derived. This bound is, in essence, a generalisation of the fact that for solid objects the surface to volume ratio goes to zero as the inverse radius. Applied to the Eden model and the related Rikvold and SOYH (Sawada *et al*) models it yields the fact that these models are compact and their perimeters are $(d-1)$-dimensional. Applied to diffusion-limited aggregates it gives support to the empirical fact that only a small minority of points on the growing interface are growing very actively and provides a simple proof of the Ball-Witten inequality for the fractal dimension of DLA. There are presumably many more such applications.

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